Section 3: REFERENCE DYNAMIC MODELS

3.1. Response of a dynamic model

A thermometer has first-order dynamics with a time constant of 1 second, steady-state gain of 1. It is placed in a temperature bath at 120°F (T_{ext}). After the thermometer reaches steady state ($T_{ext}=T_m=120$°F), it is suddenly placed in a bath at 140 °F for $0 \leq t \leq 10$ s. Then T_{ext} is returned to the bath at 100 °F.

$$T_{ext}(t) = 120 + 20\,u(t) - 40\,u(t-10)$$

Where $u(t)$ is the Heaviside step function.

1. Write the forcing function $T_{ext}(t)$ in terms of deviation variable(s) $T'_{ext}(t)$.
2. Write the forcing function $T'_{ext}(t)$ in the Laplace domain ($\hat{T}_{ext}(s)$).
3. Obtain the expression, in the Laplace domain, of the measured temperature $\hat{T}_m(s)$.
4. Obtain the expression of the time evolution of the measured temperature in terms of deviation variable $T'_m(t)$
5. Calculate the measured temperature at:
    a. t=0.5 s;
    b. t=10.5 s;
    c. t=15 s.

x	*exp(-x)*
0	1.00
0.25	0.78
0.5	0.60
0.75	0.47
1	0.37
1.5	0.22
2	0.14
2.5	0.08
3	0.05
3.5	0.03
4	0.02
≥ 4.5	0

Hints:

- The $\mathcal{L}^{-1}\{e^{-t_d s} f(s)\} = u(t-t_d) f(t-t_d)$ where $u(t)$ is the Heaviside step function.
- The provided table can be used to approximate the exponential decay function:

Solution

The steady state temperature is: $T_{ext_{ss}} = T_{m_{ss}} = 120°F$

1. The deviation variable of the forcing function $T_{ext}(t)$ is:

$$T'_{ext}(t) = T_{ext}(t) - T_{ext_{SS}}$$

$$T'_{ext}(t) = \left(120 + 20\,u(t) - 40\,u(t-10)\right) - 120$$

$$T'_{ext}(t) = 20\,u(t) - 40\,u(t-10)$$

2. The Laplace transform of $T'_{ext}(t)$ is:

$$\mathcal{L}\{T'_{ext}(t)\} = \hat{T}_{ext}(s) = 20\mathcal{L}\{u(t)\} - 40\,\mathcal{L}\{u(t-10)\}$$

$$\hat{T}_{ext}(s) = \frac{20}{s} - \frac{40}{s}\exp(-10s)$$

3. At this point the definition of transfer function has to be used. A "*first-order dynamics with a time constant of 1 second, steady-state gain of 1*" has the form:

$$G(s) = \frac{\hat{T}_m(s)}{\hat{T}_{ext}(s)} = \frac{1}{s+1}$$

Therefore:

$$\hat{T}_m(s) = G(s)\,\hat{T}_{ext}(s)$$

$$\hat{T}_m(s) = \frac{1}{s+1}\frac{20}{s} - \frac{1}{s+1}\frac{40}{s}\exp(-10s)$$

4. To obtain the $T'_m(t)$ the inverse Laplace transform has to be performed on $\hat{T}_m(s)$.

$$\mathcal{L}^{-1}\{\hat{T}_m(s)\} = \mathcal{L}^{-1}\left\{\frac{20}{s(s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{40}{s(s+1)}\exp(-10s)\right\}$$

From the "Table of Laplace Transforms" in the Reference Tables for Students [1]:

2.13	$\frac{\alpha}{s(s+\alpha)}$	$1-\exp(-\alpha t)$

$$\mathcal{L}^{-1}\left\{\frac{20}{s(s+1)}\right\} = 20(1-\exp(-t))$$

$$\mathcal{L}^{-1}\left\{\frac{40}{s(s+1)}\exp(-10s)\right\} = \cdots$$

$$\mathcal{L}^{-1}\{e^{-t_d s}F(s)\} = u(t-t_d)f(t-t_d)$$

[1] The same result can be obtained with the partial fraction decomposition.

$$\mathcal{L}^{-1}\left\{\frac{40}{s(s+1)}\exp(-10s)\right\} = 40\big(u(t-10)f(t-10)\big)$$

Who is $f(t)$?

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = (1-\exp(-t))$$

To obtain $f(t-10)$ we have to "substitute" "t" with "$t-10$" :

$$f(t-10) = 1-\exp\big(-(t-10)\big) = 1-\exp(-t+10)$$

Therefore:

$$\mathcal{L}^{-1}\left\{\frac{40}{s(s+1)}\exp(-10s)\right\} = 40\big(u(t-10)(1-\exp(10-t)\big)$$

The inverse Laplace transform of $\hat{T}_m(s)$ is:

$$\mathcal{L}^{-1}\{\hat{T}_m(s)\} = T'_m(t) = 20(1-\exp(-t)) - 40u(t-10)(1-\exp(10-t)$$

5. The measured variable is: $T_m(t) = 120 + T'_m(t)$

a. t=0.5 s

$$T_m(0.5) = 120 + 20(1-\exp(-0.5)) - 40\,u(0.5-10)(1-\exp(10-0.5))$$

The Heaviside step function is defined as:

$$u(x) = \begin{cases} 0, x \le 0 \\ 1, x > 0 \end{cases}$$

Therefore:

$$T_m(0.5) = 120 + \ 20(1-\exp(-0.5)) \sim from\ table = 120 + \ 20(1-0.6) = 128\ °F$$

b. t=10.5 s

$$T_m(10.5) = 120 + 20(1-\exp(-10.5)) - 40\,u(10.5-10)(1-\exp(10-10.5)) =$$
$$= 120 + 20(1-\exp(-10.5)) - 40\,u(0.5)(1-\exp(-0.5)) \sim from\ table =$$
$$= 120 + 20(1-0) - 40\ \times 1 \times (1-0.6) = 124°F$$

b. t=15 s

$$T_m(15) = 120 + 20(1-\exp(-15)) - 40\,u(15-10)(1-\exp(10-15)) \sim from\ table$$
$$= 120 + 20 - 40 = 100°F$$