Section 3: REFERENCE DYNAMIC MODELS

3.1. Response of a dynamic model

A chemical reaction is taking place in a tank, and the concentration of a reactant is being monitored by a concentration analyzer. The relationship between the measured concentration $C'_m(s)$ and the actual concentration $C'(s)$ is given by the following transfer function (in deviation variable form):

$$\frac{C'_m(s)}{C'(s)} = \frac{1}{s+1}$$

The system is at its steady-state (SS) value, with actual and measured concentration of 2 mol/L: C_{ss}= $C_{m_{ss}}$= 2 mol/L. A warning light on the analyzer turns on whenever the measured concentration drops below 1 mol/L. Suppose that at time t = 0 min, the concentration of the reactant in the tank begins to decrease, $C(t) = 2 - 0.5t$, where C has units of mol/L and t has units of minutes.

1. Which type of reference dynamic model is actually represented by the above transfer function?
2. How much is the steady-state gain?
3. How much is the time constant?
4. Is the process affected by delay? If so, how much is the time delay?
5. Write the forcing function $C(t)$ in terms of deviation variable(s) $C'^{(t)}$.
6. Write the forcing function $C'^{(t)}$ in the Laplace domain ($\hat{C}(s)$).
7. Obtain the expression, in the Laplace domain, of the measured concentration $\hat{C}_m(s)$.
8. Obtain the expression of the time evolution of the measured concentration in terms of deviation variable C'_m(t)
9. At what time, with the approximation of ±1 min, will the warning light turn on?

Hints:

- The following table can be used to approximate the exponential decay function:

t	*exp(-t)*
0	1.00
0.25	0.78
0.5	0.61
0.75	0.47
1	0.37
1.5	0.22
2	0.14
2.5	0.08
3	0.05
3.5	0.03
4	0.02
4.5	0.01

1) The transfer function represents a first-order system.
2) The steady-state gain is 1 (@s->0)
3) The time constant is 1.
4) The process is not affected by delay.
5) C(t) = 2 - 0.5t can be written in deviation variable form as C'(t) = -0.5t.
6) Taking the Laplace transform of $C'(t)$, we get $C'(s) = -\frac{0.5}{s^2}$.
7) Multiplying the transfer function by $C'(s)$, we get $C'_m(s) = \frac{1}{s+1} C'(s) = -\frac{1}{s+1} \cdot \frac{0.5}{s^2}$
8) Inverse Laplace transforming: $C'_m(t) = \mathcal{L}^{-1}\{C'_m(s)\} = \mathcal{L}^{-1}\left\{\frac{-0.5}{s^2(s+1)}\right\} = -0.5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$

   Eq. 2.28 "Table of Laplace Transforms" in "Reference Tables for students"

2.28	$\frac{\alpha}{s^2(s+\alpha)}$	$t - \frac{1}{\alpha}[1 - \exp(-\alpha t)]$

$$C'_m(t) = -0.5\{t - [1 - \exp(-t)]\} =$$
$$= \frac{1}{2} - \frac{\exp(-t)}{2} - \frac{t}{2}$$

9) The warning light turns on when $C_m(t) < 1$.
   Remember $C_m = C'_m + C_{m_{ss}}$.

$$C_m = 2 + \left(\frac{1}{2} - \frac{\exp(-t)}{2} - \frac{t}{2}\right)$$

   We can solve for the time t such that $C_m(t) = 1$, which gives t = 2.9 min ≈3min.
   Using the Hints (without a numerical solver) we can proceed by trial and error:
   t=2 min

$$C_m = 2 + \left(\frac{1}{2} - \frac{\exp(-2)}{2} - \frac{2}{2}\right) = 2 + \left(\frac{1}{2} - \frac{0.14}{2} - 1\right) \sim 1.4$$

   t=3 min

$$C_m = 2 + \left(\frac{1}{2} - \frac{\exp(-3)}{2} - \frac{3}{2}\right) = 2 + \left(\frac{1}{2} - \frac{0.05}{2} - 1.5\right) \sim 0.975$$

   At t=3 min the value of C_m is lower than 1, the warning light will turn on.