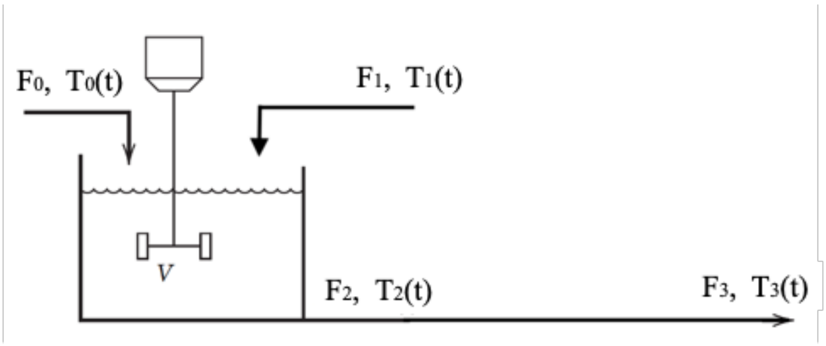
Section 6: MATHEMATICAL MODELLING OF A LUMPED PARAMETER SYSTEM

A perfectly stirred, constant-volume tank has two input streams, both consisting of the same liquid. The temperature of both the inlet streams can vary with time, whereas the volumetric flow rates F_0, F_1 and F_2 are constant. The thermal-physical properties can be assumed constant, and the output line is relatively short, so that a negligible time delay for this line can be expected $\left(T_3(t) = T_2(t)\right)$.



You must

1. write the **dynamical model** of the system;
2. write the **steady state** model of the system;
3. list **input**, **state**, **output** variables and the **parameters** of the model;
4. is the dynamical model a linear model? If not, **individuate and indicate the non-linear terms**.
5. **write the model in the Laplace domain;**
6. **obtain the transfer functions** describing the relation between the input and output variables;
7. **classify the obtained transfer functions** and individuate the parameters.

A maintenance intervention on the plant modifies the path of the output line, increasing its length. Therefore, the assumption of negligible time delay t_d for the output line does not hold anymore and it is $\left(T_3(t) \neq T_2(t)\right)$.

In such a new situation you must:

8. Write the **dynamical model** of the system;
9. **write the model in the Laplace domain**;
10. **obtain the transfer functions** describing the relation between the input and output variables.

1. ACC=IN-OUT+~~GEN~~

$$\rho V C_p \frac{dT_2(t)}{dt} = \rho F_0 C_p\left(T_0(t) - T_{ref}\right) + \rho F_1 C_p\left(T_1(t) - T_{ref}\right) - \rho F_2 C_p\left(T_2(t) - T_{ref}\right)$$

2.

$$0 = \rho F_0 C_p\left(T_{0_{ss}} - T_{ref}\right) + \rho F_1 C_p\left(T_{1_{ss}} - T_{ref}\right) - \rho F_2 C_p\left(T_{2_{ss}} - T_{ref}\right)$$

3. Input: $T_0(t), T_1(t)$
   State: $T_2(t)$
   Output: $T_2(t)$
   Parameters: $\rho, V, C_p, F_0, F_1, F_2$

4. The model is linear.
5. To rewrite the model in Laplace domain, the model in deviation variables has to be obtained:

$$\cancel{\rho} V \cancel{C_p} \frac{dT_2(t)}{dt} - 0$$
$$= \left[\cancel{\rho} F_0 \cancel{C_p}\left(T_0(t) - T_{ref}\right) - \cancel{\rho} F_0 \cancel{C_p}\left(T_{0_{ss}} - T_{ref}\right)\right]$$
$$+ \left[\cancel{\rho} F_1 \cancel{C_p}\left(T_1(t) - T_{ref}\right) - \cancel{\rho} F_1 \cancel{C_p}\left(T_{1_{ss}} - T_{ref}\right)\right]$$
$$- \left[\cancel{\rho} F_2 \cancel{C_p}\left(T_2(t) - T_{ref}\right) - \cancel{\rho} F_2 \cancel{C_p}\left(T_{2_{ss}} - T_{ref}\right)\right]$$

$$V \frac{dT_2'(t)}{dt} = F_0 T_0'(t) + F_1 T_1'(t) - F_2 T_2'(t)$$

$$\mathcal{L}\left\{V \frac{dT_2'(t)}{dt} + F_2 T_2'(t) = F_0 T_0'(t) + F_1 T_1'(t)\right\}$$

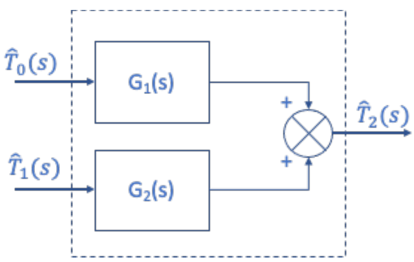
$$V \hat{T}_2 s + F_2 \hat{T}_2 = F_0 \hat{T}_0 + F_1 \hat{T}_1$$

6.

$$\hat{T}_2 (V\, s + F_2) = F_0 \hat{T}_0 + F_1 \hat{T}_1$$

$$\hat{T}_2 = \frac{F_0}{V\, s + F_2} \hat{T}_0 + \frac{F_1}{V\, s + F_2} \hat{T}_1$$

$$\hat{T}_2 = \frac{\frac{F_0}{F_2}}{\frac{V}{F_2} s + 1} \hat{T}_0 + \frac{\frac{F_1}{F_2}}{\frac{V}{F_2} s + 1} \hat{T}_1$$



$$G_1(s) = \frac{\hat{T}_2}{\hat{T}_0} = \frac{\frac{F_0}{F_2}}{\frac{V}{F_2} s + 1} = \frac{K_{P_1}}{\tau\, s + 1}$$

$$G_2(s) = \frac{\hat{T}_2}{\hat{T}_1} = \frac{\frac{F_1}{F_2}}{\frac{V}{F_2}s+1} = \frac{K_{P_2}}{\tau\, s+1}$$

7. Both $G_1(s)$ and $G_2(s)$ are first order transfer function. The parameters are $K_{P_1} = F_0/F_2$, $K_{P_2} = F_1/F_2$, $\tau = V/F_2$.

8. Energy balance on the tank and relationship between delayed variables (note that $F_3 = F_2 = constant$)):

$$\begin{cases} \rho V C_p \dfrac{dT_2(t)}{dt} = \rho F_0 C_p\left(T_0(t) - T_{ref}\right) + \rho F_1 C_p\left(T_1(t) - T_{ref}\right) - \rho F_3 C_p\left(T_2(t) - T_{ref}\right) \\ T_3(t) = T_2(t - t_D) \end{cases}$$

9. $$\mathcal{L}\begin{cases} \rho V C_p \frac{dT_2(t)}{dt} = \rho F_0 C_p\left(T_0(t) - T_{ref}\right) + \rho F_1 C_p\left(T_1(t) - T_{ref}\right) - \rho F_3 C_p\left(T_2(t) - T_{ref}\right) \\ T_3(t) = T_2(t - t_D) \end{cases}$$

$$\begin{cases} \hat{T}_2 = \dfrac{K_{P_1}}{\tau\, s+1}\hat{T}_0 + \dfrac{K_{P_2}}{\tau\, s+1}\hat{T}_1 \\ \hat{T}_3 = \hat{T}_2 \exp(-\mathrm{t_D}\, s) \end{cases}$$

10.

$$\hat{T}_3 = \frac{K_{P_1}\exp(-\mathrm{t_D}\, s)}{\tau\, s+1}\hat{T}_0 + \frac{K_{P_2}\exp(-\mathrm{t_D}\, s)}{\tau\, s+1}\hat{T}_1$$

$$G_1(s) = \frac{\hat{T}_3}{\hat{T}_0} = \frac{K_{P_1}}{\tau\, s+1}\exp(-\mathrm{t_D}\, s)$$

$$G_2(s) = \frac{\hat{T}_3}{\hat{T}_1} = \frac{K_{P_2}}{\tau\, s+1}\exp(-\mathrm{t_D}\, s)$$