Models based on population balance approach

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Population Balance models

The population balance equation (PBE) is a statement of continuity for particulate systems, originally derived in 1964.

Population balances are of key relevance to a very diverse group of scientists, including astrophysicists, high-energy physicists, geophysicists, colloid chemists, biophysicists, materials scientists, chemical engineers, and meteorologists. Chemical engineers have put population balances to most use, with applications in the areas of crystallization; gas-liquid, liquid-liquid, and solid-liquid dispersions; liquid membrane systems; fluidized bed reactors; aerosol reactors; and microbial cultures.

Engineers encounter particles in a variety of systems. The particles are either naturally present or engineered into these systems. In either case these particles often significantly affect the behavior of such systems.

This modeling approach provides a framework for analyzing these dispersed phase systems and describes how to synthesize the behavior of the population particles and their environment from the behavior of single particles in their local environments.
Population Balance models

Ramkrishna provides a clear and general treatment of population balances with emphasis on their wide range of applicability. New insight into population balance models incorporating random particle growth, dynamic morphological structure, and complex multivariate formulations with a clear exposition of their mathematical derivation is presented. Population Balances provides the only available treatment of the solution of inverse problems essential for identification of population balance models for breakage and aggregation processes, particle nucleation, growth processes, and more. This book is especially useful for process engineers interested in the simulation and control of particulate systems. Additionally, comprehensive treatment of the stochastic formulation of small systems provides for the modeling of stochastic systems with promising new areas of applications such as the design of sterilization systems and radiation treatment of cancerous tumors.

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Population Balances

Formulation

The Population Balance Equation was originally derived in 1964, when two groups of researchers studying crystal nucleation and growth recognized that many problems involving change in particulate systems could not be handled within the framework of the conventional conservation equations only, see Hulburt & Katz (1964) and Randolph (1964). They proposed the use of an equation for the continuity of particulate numbers, termed population balance equation, as a basis for describing the behavior of such systems.

This balance is developed from the general conservation equation:

\[
\text{Input - Output + Net Generation} = \text{Accumulation}
\]
Elements necessary for a population balance Model

<table>
<thead>
<tr>
<th>Element</th>
<th>General description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENTITY</td>
<td>• specific of each “particulate system”; • crystallization, biochemical processes, polymerization, leaching, comminution, and aerosols are just some examples; • entities are “discrete” and not continuous; • entities are “countable”, but may be infinite in number</td>
<td>Yeast cell in a fermenter</td>
</tr>
<tr>
<td>PROPERTIES of ENTITY</td>
<td>• in general, entities have a specified set of ( m ) properties ( \zeta_i ) with ( i = 1,2,...,m ) • the properties ( \zeta_i ) will depend on the application; • typical examples are the entity’s size diameter, entity’s chemical composition, entity’s age</td>
<td>Cell mass, cell age, etc.</td>
</tr>
<tr>
<td>ENTITY DISTRIBUTION</td>
<td>• the function ( \psi(t, x, y, z, \zeta_1, \zeta_2, ..., \zeta_m) ) represents the entity distribution • ( t ) is time • ( x, y, z ) are the spatial coordinates • it is similar to a probability density function</td>
<td>Number of cells per unit fermenter volume, unit cell mass, unit cell age, etc.</td>
</tr>
</tbody>
</table>

Entity distribution function

Physical meaning
\[
\psi(x, y, z, \zeta_1, ..., \zeta_m, t)dx, dy, dz, d\zeta_1, ..., d\zeta_m
\]
is the fraction of entities at time \( t \) that are:
- contained in the infinitesimal volume \( dV=dx dy dz \)
- characterized by values of properties in the ranges \( \zeta_1, \zeta_1+\Delta\zeta_1, \ldots, \zeta_m, \zeta_m+\Delta\zeta_m \)

Congruence constraint
\[
\int_\Omega \psi(x, y, z, \zeta_1, ..., \zeta_m, t)dx, dy, dz, d\zeta_1, ..., d\zeta_m = 1
\]
where \( \Omega \) is the (3+\( m \))-dimensional space of the independent variables:
- 3 spatial coordinates \((x, y, z)\)
- \( m \) properties \((\zeta_i \text{ with } i = 1,2,...,m)\)

\( \textcircled{1} \) time \( t \) may be one additional independent variable
Net Generation

Generation (Positive and Negative) terms are defined as:

\[ N = \frac{\text{Entity born}}{(\text{time unit})(\text{volume unit})(\text{unitary variation of the } "i" \text{ property})} \]

\[ M = \frac{\text{Entity disappearance}}{(\text{time unit})(\text{volume unit})(\text{unitary variation of the } "i" \text{ property})} \]

HYPOTHESES:

- large, arbitrary, time-varying sub-space \( R(t) \subseteq \Omega \)
- closed sub-space, with no input or output of entities
- a balance can be written in the usual way:

\[
\text{Accumulation} = \text{Net Generation} = \frac{d}{dt}\int_{R(t)} f \, dR = \int_{R(t)} (N - M) \, dR \quad (I)
\]

\[ dR = dx dy dz \zeta_1 \ldots d\zeta_m \]

Leibnitz formula

One dimension

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) \, dx = \int_{a(t)}^{b(t)} \left[ \frac{\partial f(x, t)}{\partial t} + \frac{\partial x}{\partial t} f(x, t) \right] \, dx
\]

Multi-dimensional

\[
\frac{d}{dt} \int_{R(t)} f \, dR = \int_{R(t)} \left[ \frac{\partial f}{\partial t} + \sum_{l} \frac{\partial}{\partial l} \left( \frac{dl}{dt} f \right) \right] \, dR = \int_{R(t)} \left[ \frac{\partial f}{\partial t} + \nabla \cdot \left( \mathbf{i} f \right) \right] \, dR
\]

where:

- \( R(t) \) is a time-variable region of the space \( \Omega \)
- \( f(\cdot) \) is a scalar function
- \( \mathbf{l} \) stands for any of the non-time variables \( x, y, z, \zeta_1, \zeta_2, \ldots \)
- the sum \( \Sigma \) is over all such variables
- \( \mathbf{i} = \) is the column vector of all \( \frac{dl}{dt} \)
Generalized population balance model on «microscopic scale»

The total derivatives with respect to time included in the Leibnitz formula can be explained in a physical way:

\[
\begin{align*}
\frac{dx}{dt} &= v_x; \\
\frac{dy}{dt} &= v_y; \\
\frac{dz}{dt} &= v_z; \\
\end{align*}
\]

\(v_i\) is the change rate with respect to the time or kinetics of the property \(\zeta_i\).

The equation (I) becomes:

\[
\int_{R(t)} \left[ \frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x} (v_x \psi) + \frac{\partial}{\partial y} (v_y \psi) + \frac{\partial}{\partial z} (v_z \psi) + \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} (v_i \psi) + M - N \right] dR = 0
\]

As the region \(R(t)\) is arbitrary, the necessary condition for that equation to be true is that the integrating term is null

\[
\frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x} (v_x \psi) + \frac{\partial}{\partial y} (v_y \psi) + \frac{\partial}{\partial z} (v_z \psi) + \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} (v_i \psi) + M - N = 0
\]

This is the generalized population balance model on «microscopic scale»

(in x, y, z coordinates)

Switch to «macroscopic scale»

Often \(\psi\) spatial dependence is not known or not desired, while only averaged value on system geometric volume \(V\) is requested.

The volume-averaged entity distribution is:

\[
\overline{\psi} = \frac{1}{V} \int \psi dV
\]

Integrating over the whole volume the generalized population balance model:

\[
\int \left[ \frac{\partial}{\partial t} \psi + \nabla \cdot (\bar{v} \psi) + \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} (v_i \psi) + (M - N) \right] dV = 0
\]

Let’s examine the various terms:

The term (I) can be obtained by an inversion of the multi-dimensional Leibnitz formula, restricted to the spatial coordinates only:

\[
\int \frac{\partial}{\partial t} \psi dV = \frac{d}{dt} \int \psi dV - \int \nabla \cdot (\bar{v} \psi) dV
\]

Where \(\bar{v}_s\) indicates the speedness of the surface \(S\) which contains the volume \(V\) translating and varying.
According to Gauss divergence theorem, the term $V$ becomes:

$$\int_{V} \nabla \cdot \left( \vec{v} \psi \right) dV = \int_{S} \vec{n} \cdot \left( \vec{v} \psi \right) dS$$

Vector $\vec{v}_s$ indicates the speedness for every point of the surface $S$ containing control volume $V$. $\vec{n}$ is the normal versor to $S$ directed toward the external.

The term $II$ is transformed by the same Gauss theorem:

$$\int_{V} \nabla \cdot \left( \vec{v} \psi \right) dV = \int_{S} \vec{n} \cdot \left( \vec{v} \psi \right) dS$$

The terms $I$ and $II$ (included $V$) can be combined together:

$$\int \left\{ \frac{\partial \psi}{\partial t} + \nabla \cdot \left( \vec{v} \psi \right) \right\} dV = \frac{d}{dt} \int_{V} \psi dV + \int_{S} \vec{n} \cdot \left( \vec{v} - \vec{v}_s \right) \psi dS$$

In the term $VI$ the surface $S$ is considered as the sum of three contributions:

$$S = S_1 + S_2 + S^*$$

Where:
- $S_1$ is inlet surface to the system
- $S_2$ is the outlet surface from the system
- $S^*$ is the remaining part of the surface (impenetrable)

The term $VI$ becomes:

$$\int \vec{n} \cdot \left( \vec{v} - \vec{v}_s \right) \psi dS =$$

$$\int_{S} \vec{n} \cdot \left( \vec{v} - \vec{v}_s \right) \psi dS + \int_{S^*} \vec{n} \cdot \left( \vec{v} - \vec{v}_s \right) \psi dS^* = -\int_{V_1} \psi dS_1 + \int_{V_2} \psi dS_2 + 0$$

Where:
- $V_1$ is the relative speedness in the inlet section (directed toward the opposite verse to the normal $\vec{n}$)
- $V_2$ is the relative speedness in the outlet section
- Term $VII$ is null because on $S^*$ $\vec{n} \cdot \left( \vec{v} - \vec{v}_s \right) = 0$
Further Hypothesis:
\( \psi_1 \) and \( \psi_2 \) are averaged and, therefore, constant, respectively on \( S_1 \) and \( S_2 \).

The terms \textbf{VIII} and \textbf{IX} can be simplified as follows:

\[
\int \nabla_1 \psi_1 dS_1 = \frac{\psi_1}{S_1} \int \nabla_1 dS_1 = \frac{\psi_1 Q_1}{S_1} ; \quad \int \nabla_2 \psi_2 dS_2 = \frac{\psi_2}{S_2} \int \nabla_2 dS_2 = \frac{\psi_2 Q_2}{S_2}
\]

where \( Q_1 \) and \( Q_2 \) are volumetric flowrates through \( S_1 \) and \( S_2 \), respectively.

The term \textbf{III} is integrated over the volume \( V \) and yields:

\[
\int \sum_{i=1}^{m} \frac{1}{\psi_i} \frac{\partial}{\partial \zeta_i} (\nabla_1 \psi_i) dV = \sum_{i=1}^{m} \int \frac{1}{\psi_i} \frac{\partial}{\partial \zeta_i} (\nabla_1 \psi_i) dV = 
\]

\[
= \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} \int \nabla_1 \psi_i dV = \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} \left[ \frac{\psi_1}{S_1} \int \nabla_1 dS_1 \right] = \frac{V}{S_1} \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} \frac{\psi_1}{S_1}
\]

The term \textbf{IV} yields:

\[
\int (M - N) dV = V \left( \frac{\overline{M}}{V} - \frac{\overline{N}}{V} \right)
\]

Population balance model on «macroscopic scale»

After replacing all the terms:

\[
V \frac{\partial}{\partial t} (\overline{\psi}) + V \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} (\nabla_1 \overline{\psi}) + V \left( \frac{\overline{M}}{V} - \frac{\overline{N}}{V} \right) - Q_1 \overline{\psi_1} + Q_2 \overline{\psi_2} = 0
\]

Dividing by \( V \):

\[
\frac{\partial}{\partial t} (\overline{\psi}) + \sum_{i=1}^{m} \frac{\partial}{\partial \zeta_i} (\nabla_1 \overline{\psi}) + \overline{M} - \overline{N} = \left[ \frac{Q_1 \overline{\psi_1} - Q_2 \overline{\psi_2}}{V} \right] \frac{1}{V}
\]

This is the population balance model on «macroscopic scale» (without \( x, y, z \) coordinates)
Example No. 1
Vapor Deposition in a gas-solid fluidized bed

Hypotheses

- Macroscopic approach
- Steady state
- Constant inventory of particles in the fluidized bed
- All particles having the same chemical composition
- Particles having spherical geometry
- Particles having constant density
- Negligible elutriation
- Negligible solid-solid and solid-wall attrition

NOMENCLATURE

- \( W \) = bed mass (inventory of particles)
- \( F_0 \) = mass feed rate of fresh particles
- \( F_1 \) = mass discharge rate of processed particles
Vapor Deposition in a gas-solid fluidized bed

Entity: solid particles
Property: particle diameter \( d \)
Objective: \( \rightarrow p(d) \equiv m^{-1} \)

\( p(d) \) is the mass fraction of particles with diameter between \( d \) and \( d+\Delta d \)

Population Balance approach: the generation term

\[ \bar{N}(d) = \frac{W \cdot p(d) \Delta d}{\rho_p \cdot \frac{\pi}{6} \cdot d^3} \cdot \frac{dV_p}{dt} \cdot \rho_p \cdot \frac{1}{\Delta d} = \frac{W \cdot p(d) \Delta d}{\rho_p \cdot \frac{\pi}{6} \cdot d^3} \cdot \frac{\pi}{3} \cdot \frac{d^2}{dt} \cdot \rho_p \cdot \frac{1}{\Delta d} = \]

\[ = \frac{3 \cdot W \cdot p(d) \cdot \mathcal{R}(d)}{d} \equiv \text{mass} \cdot \frac{1}{(\text{time}) \cdot (\text{unit property change})} \]

\( \mathcal{R}(d) \) is the rate of particle size change with time (property kinetics)
The overall generation

\[ \bar{N}_{tot} = \int_{0}^{\infty} \bar{N}(d) \, dd = \int_{0}^{\infty} 3 \cdot W \cdot p(d) \cdot \mathcal{R}(d) \, dd \quad \text{[generated mass]} \]

\[ \text{time} \]

Overall Mass Balances

on the solid phase with ref. to the whole fluid bed:

1. \[ F_0 - F_1 + \bar{N}_{tot} = 0 \]

\( \text{⊕} \) This is NOT a Population Balance eq.

on the gas phase with ref. to the whole fluid bed:

1 bis. \[ \dot{m}_{IN} - \dot{m}_{OUT} - \bar{N}_{tot} = 0 \]
Population Balance approach: the mass balance

per unit time and with ref. to the size interval \([d; d+dd]\):

\[
F_0 \cdot p_0(d)dd - F_1 \cdot p_1(d)dd + \left[ W \cdot p(d) \cdot \frac{dd}{dt} \right]_d - \left[ W \cdot p(d) \cdot \frac{dd}{dt} \right]_{d+dd} + \frac{3 \cdot W \cdot p(d) \cdot R(d)dd}{d} = 0
\]

Further Hypotheses

✓ Back mixing: \(p_1(d) = p(d)\)

✓ “Monosize” feed: \(p_0(d) = \delta(d-D_0)\)

\[
\delta(d)=0 \text{ for } d \neq D_0
\]

\[
\delta=\infty \text{ for } d=D_0
\]

\[
\int_0^\infty \delta(d-D_0)dd = 1
\]

with ref. to an infinitesimal size interval located immediately below the feed size \( D_0 \): \([D_0; D_0-\delta d]\)

\[
F_0 \cdot p_0(d) - F_1 \cdot p(d) - W \cdot \mathcal{R}(d) \frac{dp(d)}{dd} - W \cdot p(d) \frac{d\mathcal{R}(d)}{dd} + 3 \cdot W \cdot p(d) \cdot \mathcal{R}(d) = 0
\]

\[
F_0 \cdot p_0(d) dd - F_1 \cdot p(d) dd - W \cdot \mathcal{R}(d) \cdot dp(d) - W \cdot p(d) dd \cdot \frac{d\mathcal{R}(d)}{dd} + 3 \cdot W \cdot \mathcal{R}(d) \cdot p(d) dd = 0
\]

\[
F_0 \cdot p_0(d) dd \frac{d\mathcal{R}(d)}{\mathcal{R}(d)} - W dp(d) = 0 \rightarrow p(d) dd = 0
\]

integration over \([0, D_0]\) gives \( p(D_0) \):

\[
\int_0^{D_0} \frac{F_0 \cdot p_0(d)}{\mathcal{R}(d)} dd - \int_0^{p(D_0)} W dp(d) = 0
\]

\[
F_0 \cdot \int_0^{D_0} \frac{p_0(d)}{\mathcal{R}(d)} dd = W \cdot \int_0^{p(D_0)} dp(d)
\]

\[
\Rightarrow \quad p_0(d) = \delta(d - D_0)
\]

\[
\Rightarrow \int_0^{D_0} \delta(d - D_0) \frac{d\mathcal{R}(d)}{\mathcal{R}(d)} dd = \frac{1}{\mathcal{R}(D_0)} 4. \quad p(D_0) = \frac{F_0}{W \cdot \mathcal{R}(D_0)} \neq 0
\]

---


with ref. to an infinitesimal size interval \([d; d+\delta d]\) located above the feed size \( D_0 \):

\[
F_0 \cdot p_0(d) dd - F_1 \cdot p(d) dd - W \cdot \mathcal{R}(d) \cdot dp(d) - W \cdot p(d) dd \cdot \frac{d\mathcal{R}(d)}{dd} + 3 \cdot W \cdot \mathcal{R}(d) \cdot p(d) dd = 0
\]

manipulation and integration over \([D_0, d]\) with \( d > D_0 \) give:

\[
- \int_{D_0}^{d} \frac{F_1}{W \cdot \mathcal{R}(d)} dd - \int_{p(D_0)}^{p(d)} \frac{dp(d)}{p(d)} - \int_{\mathcal{R}(D_0)}^{\mathcal{R}(d)} \frac{d\mathcal{R}(d)}{\mathcal{R}(d)} + 3 \cdot \int_{D_0}^{d} \frac{dd}{d} = 0
\]

\[
- \frac{F_1}{W} \int_{D_0}^{d} \frac{dd}{\mathcal{R}(d)} + \ln \frac{p(D_0)}{p(d)} + \ln \frac{\mathcal{R}(D_0)}{\mathcal{R}(d)} + 3 \cdot \ln \frac{d}{D_0} = 0
\]
Taking the exponential, replacing eq. 4 and then solving for \( p(d) \) gives:

\[
\text{Population Balance approach: entity distribution eq. } p(d) = \frac{F_0}{W \cdot \mathcal{R}(D_0)} \left( \frac{\int_{D_0}^{d} dd' \cdot \mathcal{R}(d')}{\int_{D_0}^{d} dd' \cdot \mathcal{R}(d')} - \frac{F_1}{W} \int_{D_0}^{d} dd' \cdot \mathcal{R}(d') \right)
\]

where \( d' \) is an integration variable.

**Population Balance approach: congruence condition for \( p(d) \)**

\[
\int_{D_0}^{\infty} p(d) dd = 1
\]

\[
W = \frac{F_0}{D_0} \cdot \int_{D_0}^{\infty} \frac{d^3}{\mathcal{R}(d)} \cdot e^{\int_{D_0}^{d} dd' \cdot \mathcal{R}(d')} dd
\]
Population Balance approach: Final equations

1. \[ F_1 - F_0 = 3W \int_{D_0}^{\infty} p(d) \cdot \mathcal{R}(d) d \] 

2. \[ W = \frac{F_0}{D_0^3} \cdot \int_{D_0}^{\infty} \frac{d^3}{\mathcal{R}(d)} \cdot e^{-\frac{F_1}{W} \int_{D_0}^{d} \frac{dd'}{\mathcal{R}(d')}} d \] 

3. \[ p(d) = \frac{F_0}{W \cdot \mathcal{R}(d)} \cdot \frac{d^3}{D_0^3} \cdot e^{-\frac{F_1}{W} \int_{D_0}^{d} \frac{dd'}{\mathcal{R}(d')}} \] 

Population Balance approach: Integro-differential equations

More generally, size distributions are time dependent. Ex.: rate of change of the distribution (shape) depends on the whole distribution. 

- Agglomeration of these may form a particle of size \( s \)
- Breakage of these may form a particle of size \( s \)
- \( x(s,t) \) under consideration
- Breakage of \( x(s,t) \) affects the distribution
**Example No. 2**

**Sublimation in a gas-solid fluidized bed**

- **Macroscopic approach**
- Batch operation (dynamical system)
- “Active” particles (i.e., sublimating particles) different from the particles constituting the fluidized bed
- Perfect mixing
- All “Active” particles maintain the same composition
- Particles having spherical geometry
- Particles having constant density
- Negligible elutriation
- Negligible solid-solid and solid-wall attrition

** NOMENCLATURE**

- \( W_0 \) = overall mass of sublimating solid particles at time 0
- \( W(t) \) = overall mass of sublimating solid particles at time \( t \)
Population Balance approach: batch sublimation in fluidized bed

**Entity:** mass of sublimating solid particles

**Property:** particle diameter $d$

**Objective:** $\rightarrow m(d, t) \ [\text{=}] \ \text{kg/m}$

$m(d, t)\Delta d$ is the mass of sublimating particles in the bed with diameter between $d$ and $d + \Delta d$

---

control volume in the "property space"

$\Delta d$

mass Balance on the size interval $\Delta d$ and in the time interval $\Delta t$

$$m(d + \Delta d, t) \frac{dd}{dt} \bigg|_{d+\Delta d} \Delta t - m(d, t) \frac{dd}{dt} \bigg|_{d} \Delta t - \frac{3m(d, t)\Delta d \mathcal{R}(d)}{d} \Delta t = \left\{ [m(d, t)]_{t+\Delta t} - [m(d, t)]_{t} \right\} \Delta d$$

where: $\mathcal{R}(d) = \frac{dd}{dt} \ [\text{=} \ \text{m/s}]$
Population Balance approach: batch sublimation in fluidized bed

Let us divide by $\Delta t$ for $\Delta t \to 0$

\[
m(d + \Delta d, t) \frac{dd}{dt} \bigg|_{d+\Delta d} - m(d, t) \frac{dd}{dt} \bigg|_{d} - \frac{3m(d, t) \Delta d \mathcal{R}(d)}{d} = \frac{\partial m(d, t)}{\partial t} \Delta d
\]

Let us divide by $\Delta d$

\[
m(d + \Delta d, t) \frac{dd}{dt} \bigg|_{d+\Delta d} - m(d, t) \frac{dd}{dt} \bigg|_{d} - \frac{3m(d, t) \mathcal{R}(d)}{d} = \frac{\partial m(d, t)}{\partial t}
\]

For $\Delta d \to 0$

\[
\frac{\partial [m(d, t) \mathcal{R}(d)]}{\partial d} - \frac{3m(d, t) \mathcal{R}(d)}{d} = \frac{\partial m(d, t)}{\partial t}
\]

PDE subject to:

IC: for $t=0$ \[m(d, 0) = m_0(d)\]

BC: for $d=0$, $\forall t$ \[m(0, t) = 0\]

Comparison with the population balance model on «macroscopic scale»

\[
\frac{\partial \psi}{\partial t} \Rightarrow \frac{\partial m(d, t)}{\partial t}
\]

\[
\sum_{i=1}^{m} \frac{\partial (v_i \psi)}{\partial \zeta_i} \Rightarrow \frac{\partial [m(d, t) \mathcal{R}(d)]}{\partial d}
\]

\[
\mathcal{M} \Rightarrow \frac{3m(d, t) \mathcal{R}(d)}{d}
\]

Total mass at time $t$:

\[
W(t) = \int_{0}^{d_{max}} m(d, t) dd
\]
Population Balance approach: batch sublimation in fluidized bed

Particular cases for sublimation kinetics

1. Volume
\[
\frac{dm_p}{dt} = \rho_p \frac{\pi dd^3}{6} = -b\pi d^3 \quad \text{[kg/s]}
\]
\[
\rho_p \frac{3\pi d^2 dd}{6} = -b\pi d^3
\]
\[
\mathcal{R}(d) = -hd
\]
\[
-hd \frac{\partial m(d, t)}{\partial d} - hm(d, t) + 3hm(d, t) = \frac{\partial m(d, t)}{\partial t}
\]

2. Surface
\[
\frac{dm_p}{dt} = \rho_p \frac{\pi dd^3}{6} = -c\pi d^2 \quad \text{[kg/s]}
\]
\[
\rho_p \frac{3\pi d^2 dd}{6} = -c\pi d^2
\]
\[
\mathcal{R}(d) = -k
\]
\[
-k \frac{\partial m(d, t)}{\partial d} + 3km(d, t) = \frac{\partial m(d, t)}{\partial t}
\]

Yeast cell distribution in a fermenter