16. Block Diagrams

16.1 Combining Transfer Functions Using Block Diagrams

Laplace domain transfer functions enable us to manipulate complex equations using simple algebra. For example, consider two non-interacting draining tanks. If we assume that drain flow rate is proportional to hydrostatic head, we can write the equations using perturbation variables:

The tank 1 ODE is
$$A_{C1}\frac{dh_1^P(t)}{dt} + \alpha_1 h_1^P(t) = F_0^P(t) \qquad \text{where } h_1^P(0) = 0 \tag{16.1}$$

In the Laplace domain, Eq. 16.1 becomes

$$A_{C1} s H_1(s) + \alpha_1 H_1(s) = F_0\,(s) \tag{16.2}$$

The tank 1 transfer function is thus
$$G_{P1}(s) = \frac{H_1(s)}{F_0(s)} = \frac{1/\alpha_1}{\left(A_{C1}/\alpha_1\right)s + 1} \tag{16.3}$$

The tank 2 ODE is
$$A_{C2}\frac{dh_2^P(t)}{dt} + \alpha_2 h_2^P(t) = \alpha_1 h_1^P(t) \qquad \text{where } h_2^P(0) = 0 \tag{16.4}$$

In the Laplace domain, Eq. 16.4 becomes

$$A_{C2} s H_2(s) + \alpha_2 H_2(s) = H_1(s) \tag{16.5}$$

The tank 2 transfer function is thus
$$G_{P2}(s) = \frac{H_2(s)}{H_1(s)} = \frac{\alpha_1/\alpha_2}{\left(A_{C2}/\alpha_2\right)s + 1} \tag{16.6}$$

With the above as a basis, we write general coupled process ODEs as

Process 1: $\tau_{P1}\dfrac{dy_1(t)}{dt} + y_1(t) = K_{P1}u(t)\,;\; y_1\,(0) = 0 \quad \Rightarrow \quad G_{P1}(s) = \dfrac{Y_1(s)}{U(s)} = \dfrac{K_{P1}}{\tau_{P1}s + 1}$ (16.7)

Process 2: $\tau_{P2}\dfrac{dy_2(t)}{dt} + y_2(t) = K_{P2}y_1(t)\,;\; y_2(0) = 0 \quad \Rightarrow \quad G_{P2}(s) = \dfrac{Y_2(s)}{Y_1(s)} = \dfrac{K_{P2}}{\tau_{P2}s + 1}$ (16.8)

Combining the time domain ODE's of Eqs. 16.7 and 16.8 into a single second-order differential equation describing how $y_2(t)$ responds to changes in $u(t)$ requires manipulation of ODEs as follows:

Solve Eq. 16.8 for $y_1(t)$:
$$y_1(t) = \frac{\tau_{P2}\dfrac{dy_2(t)}{dt} + y_2(t)}{K_{P2}} \tag{16.9}$$

Take the derivative of Eq. 16.9:

$$\frac{dy_1(t)}{dt} = \frac{\tau_{P2}\dfrac{d^2 y_2(t)}{dt^2} + \dfrac{dy_2(t)}{dt}}{K_{P2}} \tag{16.10}$$

Substitute Eqs. 16.9 and 16.10 into Eq. 16.7:

$$\tau_{P1}\frac{\tau_{P2}\dfrac{d^2 y_2(t)}{dt^2} + \dfrac{dy_2(t)}{dt}}{K_{P2}} + \frac{\tau_{P2}\dfrac{dy_2(t)}{dt} + y_2(t)}{K_{P2}} = K_{P1}u(t) \tag{16.11}$$

Multiply both sides by K_{P2} and combine like terms:

$$\boxed{\tau_{P1}\tau_{P2}\frac{d^2 y_2(t)}{dt^2} + (\tau_{P1} + \tau_{P2})\frac{dy_2(t)}{dt} + y_2(t) = K_{P1}K_{P2}u(t)} \tag{16.12}$$

In the Laplace domain, we combine the transfer functions using simple algebra, which is the reason for converting from the time domain into the Laplace domain and back:

$$G_{\text{system}}(s) = G_{P1}(s)G_{P2}(s) = \frac{\cancel{Y_1(s)}}{U(s)}\frac{Y_2(s)}{\cancel{Y_1(s)}} = \frac{Y_2(s)}{U(s)} = \left(\frac{K_{P1}}{\tau_{P1}s+1}\right)\left(\frac{K_{P2}}{\tau_{P2}s+1}\right) \tag{16.13}$$

and thus

$$\frac{Y_2(s)}{U(s)} = \frac{K_{P1}K_{P2}}{\tau_{P1}\tau_{P2}s^2 + (\tau_{P1} + \tau_{P2})s + 1} \tag{16.14}$$

As we expect, converting Eq. 16.14 back to the time domain yields Eq. 16.12. This comparison of time domain versus Laplace domain equation manipulation helps demonstrate the benefit of using the Laplace domain in our subsequent analyses.

A block diagram is a convenient way to visualize the combination and manipulation of Laplace domain equations. As shown in Fig. 16.1, we use a summer block (a circle) to add block inputs and a multiplier block (a square) to multiply block inputs:

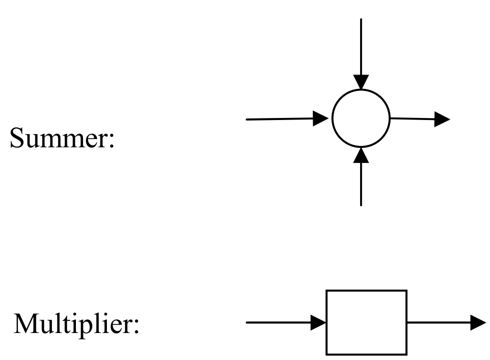
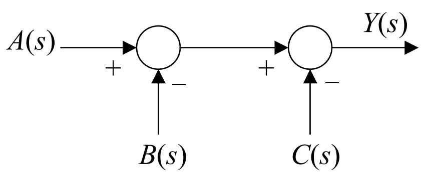


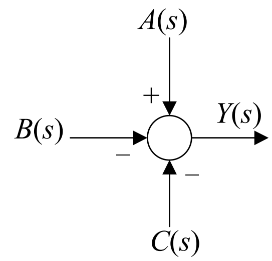
Figure 16.1 – Blocks used to create a Laplace domain block diagram

Example 1: Show this manipulation using block diagrams: $Y(s) = A(s) - B(s) - C(s)$

Solution: Below are two of several possibilities:



$[A(s) - B(s)] - C(s) = Y(s)$



$A(s) - B(s) - C(s) = Y(s)$

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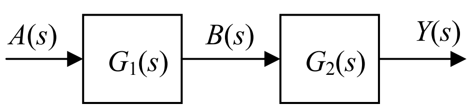
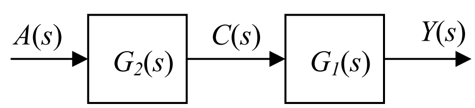
Example 2: Show this manipulation using block diagrams: $Y(s) = A(s)G_1(s)G_2(s)$

Solution: Below are two of several possibilities:



$Y(s) = B(s)G_2(s)$
$B(s) = A(s)G_1(s)$
$Y(s) = A(s)G_1(s)G_2(s)$



$Y(s) = C(s)G_1(s)$
$C(s) = A(s)G_2(s)$
$Y(s) = A(s)G_1(s)G_2(s)$

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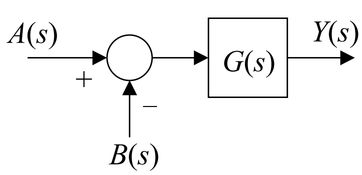
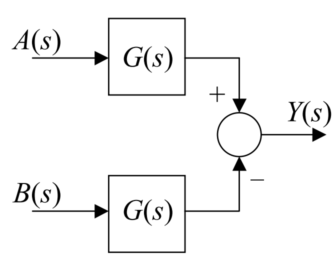
Example 3: Show using block diagrams: $Y(s) = [A(s) - B(s)]G(s)$

Solution: Below are two of several possibilities:



$Y(s) = [A(s) - B(s)]G(s)$



$Y(s) = A(s)G(s) - B(s)G(s) = [A(s) - B(s)]G(s)$

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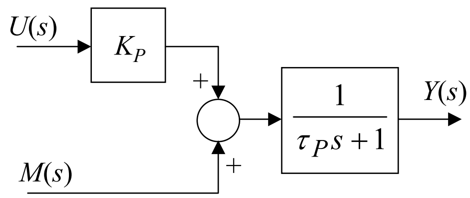
Example 4: Show using block diagrams: $Y(s) = \frac{K_P}{\tau_P s+1}U(s) + \frac{1}{\tau_P s+1}M(s)$

Solution: Below is one of several possibilities:



$$Y(s) = [U(s)K_P + M(s)]\left[\frac{1}{\tau_P s+1}\right] = \frac{K_P}{\tau_P s+1}U(s) + \frac{1}{\tau_P s+1}M(s)$$

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16.2 The Closed Loop Block Diagram

As shown in Fig. 16.2, the closed loop block diagram in the time domain is

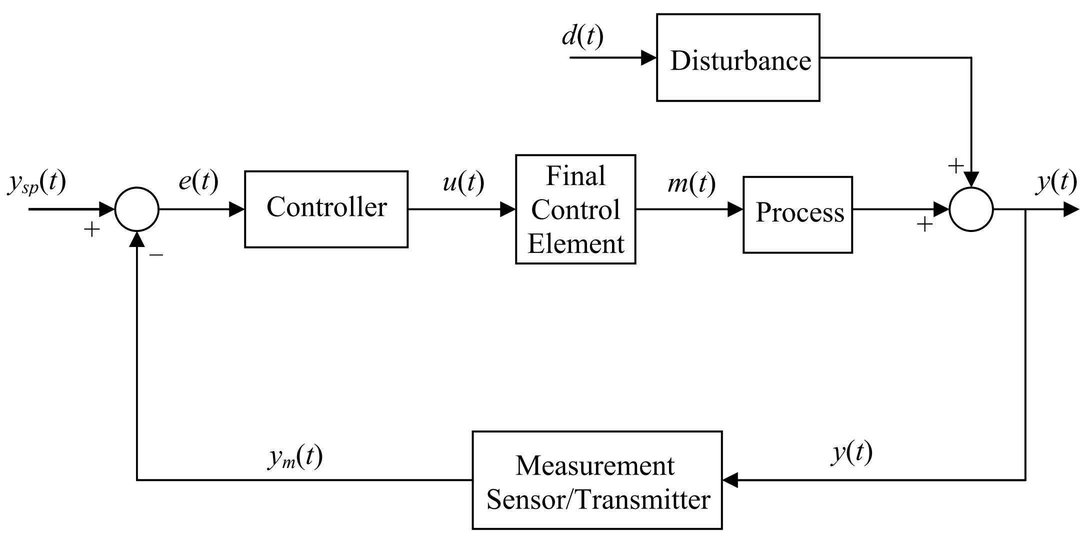

Figure 16.2 – Closed Loop Block Diagram in Time Domain

In the Laplace domain, the closed loop block diagram is as shown in Fig. 16.3:

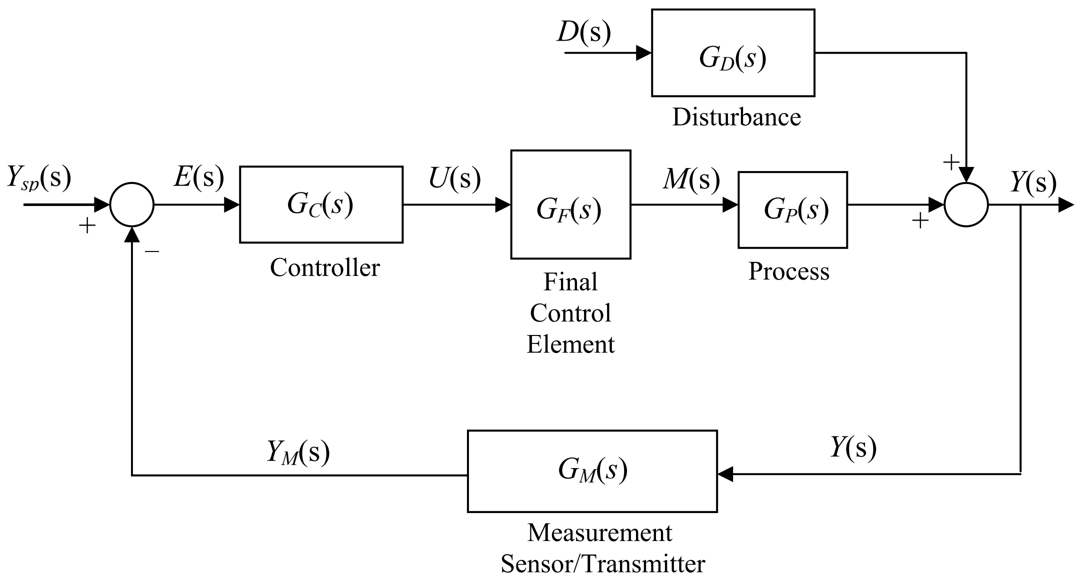

Figure 16.3 – Closed Loop Block Diagram in Laplace Domain

Notice that it is not only the process and controller that have transfer functions describing their dynamic behavior. As shown in the block diagram of Fig. 16.3, final control element (e.g. valve, pump) and measurement sensor also have transfer functions:

$$G_C(s) = \frac{U(s)}{E(s)} \qquad G_F(s) = \frac{M(s)}{U(s)} \qquad G_M(s) = \frac{Y_M(s)}{Y(s)} \tag{16.15}$$

controller — final control element — measurement sensor

The block diagram shows that:

$$E(s) = Y_{sp}(s) - Y_M(s) \tag{16.16}$$

The block diagram also shows that the transfer function for the measured process variable is somewhat more complicated when a disturbance variable is included. Following the block diagram rules presented above, the transfer function is:

$$Y(s) = M(s)G_P(s) + D(s)G_D(s) \tag{16.17}$$

16.3 Closed Loop Block Diagram Analysis

Building on the principles discussed earlier in this chapter, we can write a series of equations as we step around the closed loop block diagram in an orderly fashion. A convenient place to start in the balance is with process variable $Y(s)$ as it exits the block diagram on the right. The equations thus develop as:

$$Y(s) = M(s)G_P(s) + D(s)G_D(s) \tag{16.18a}$$

$$M(s) = U(s)G_F(s) \tag{16.18b}$$

$$U(s) = E(s)G_C(s) = [Y_{sp}(s) - Y_M(s)]G_C(s) \tag{16.18c}$$

$$Y_M(s) = Y(s)G_M(s) \tag{16.18d}$$

Substituting Eq. 16.18b into Eq. 16.18a, and Eq. 16.18d into Eq. 16.18c yields:

$$Y(s) = U(s)G_F(s)G_P(s) + D(s)G_D(s) \tag{16.19a}$$

$$U(s) = [Y_{sp}(s) - Y(s)G_M(s)]G_C(s) \tag{16.19b}$$

Substituting Eq. 16.19b into Eq. 16.19a yields:

$$Y(s) = [Y_{sp}(s) - Y(s)G_M(s)]G_C(s)G_F(s)G_P(s) + D(s)G_D(s) \tag{16.20}$$

$$= Y_{sp}(s)G_C(s)G_F(s)G_P(s) - Y(s)G_M(s)G_C(s)G_F(s)G_P(s) + D(s)G_D(s) \tag{16.21}$$

Rearrange to obtain

$$Y(s)[1+ G_M(s)G_C(s)G_F(s)G_P(s)] = Y_{sp}(s)G_C(s)G_F(s)G_P(s) + D(s)G_D(s) \tag{16.22}$$

Combining these equations and solving for $Y(s)$ produces the following closed loop Laplace equation:

$$Y(s) = \frac{G_C(s)G_F(s)G_P(s)}{1+G_C(s)G_F(s)G_P(s)G_M(s)} Y_{sp}(s) + \frac{G_D(s)}{1+G_C(s)G_F(s)G_P(s)G_M(s)} D(s) \tag{16.23}$$

Here we realize that a complex transfer function can be constructed from a combination of simpler transfer functions. As this analysis reveals, the closed loop transfer functions are

Process Variable to Set Point (when disturbance is constant):

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{G_C(s)G_F(s)G_P(s)}{1+G_C(s)G_F(s)G_P(s)G_M(s)}$$

Process Variable to Disturbance (when set point is constant):

$$\frac{Y(s)}{D(s)} = \frac{G_D(s)}{1+G_C(s)G_F(s)G_P(s)G_M(s)}$$

With the controller in automatic (closed loop), if the dynamics are disturbance driven or set point driven, the characteristic equation that reveals the inherent dynamic character of the system is the denominator of the transfer function, which in this case is

$$1+G_C(s)G_F(s)G_P(s)G_M(s) = 0 \tag{16.24}$$

Recall that the roots of the characteristic equation (the poles of the transfer function) indicate whether or not a system is stable and the degree to which it has tendency to oscillate. The analysis above reveals that the roots of Eq. 16.24 will provide this same important information for a closed-loop control system.

16.4 Simplified Block Diagram

While the final control element, process and sensor/transmitter have individual dynamics, from a controller's viewpoint it is impossible to separate these different behaviors. A controller sends a signal out on one wire and sees the response in the process variable when the measurement returns on another wire. As a consequence, the individual gains, time constants and dead times all lump together into a single overall dynamic response. A lumped or simplified block diagram can represent this as:

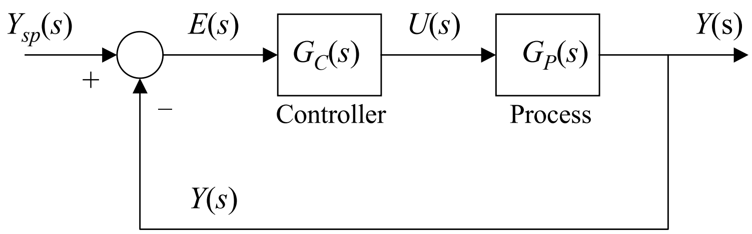

Figure 16.3 – Simplified Closed Loop Block Diagram in Laplace Domain

As before, we write a balance around the closed loop block diagram of Fig. 16.3 that starts and ends with $Y(s)$:

$$Y(s) = U(s)G_P(s)$$

$$U(s) = E(s)G_C(s) = [Y_{sp}(s) - Y(s)]G_C(s)$$

Combining these equations and solving for $Y(s)$ produces the *closed-loop process variable to set point transfer function* that describes the dynamic response of the measured process variable in response to changes in set point:

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)} \quad (16.25)$$

The characteristic equation for this closed-loop system is the denominator of the transfer function of Eq. 16.25, or:

$$1 + G_C(s)G_P(s) = 0 \quad (16.26)$$

The roots of Eq. 16.26, which are the poles of the transfer function of Eq. 16.25, indicate whether or not the closed-loop system is stable and the degree to which it has tendency to oscillate.

16.5 The Padé Approximation

Before we continue with our analysis of block diagrams, we recognize the need for a rational expression for dead time in the Laplace domain, $e^{-\theta s}$. This will permit us to employ normal algebraic manipulations during our analysis. The Taylor series expansion for $e^{-\theta s}$ is:

$$e^{-\theta s} = 1-\theta s+\frac{\theta^2 s^2}{2!}-\frac{\theta^3 s^3}{3!}+\frac{\theta^4 s^4}{4!}+\ \ldots \qquad (16.27)$$

For very small values of dead time we can truncate the series as:

$$e^{-\theta s} \cong 1-\theta s \qquad (16.28)$$

A Padé approximation is a clever expression that more accurately approximates the Taylor series of Eq. 16.27 while providing the rational expression we seek. There are a family of Padé expressions that become increasingly accurate as they increase in complexity. A simple Padé form we use in the next section is exact for the first three terms of the Taylor series expansion and quite close for the fourth term:

$$e^{-\theta s} \cong \frac{2-\theta s}{2+\theta s}$$

which we can show using long division yields the series:

$$\frac{2-\theta s}{2+\theta s} = 1-\theta s+\frac{\theta^2 s^2}{2}-\frac{\theta^3 s^3}{4}+\ \ldots$$

16.6 Closed Loop Analysis Using Root Locus

The poles of interest for our simplified closed loop system are the roots of Eq. 16.26:

$$1+G_C(s)G_P(s)=0$$

This analysis assumes that the process behavior, and thus, the process transfer function, remains constant. Adjustable controller tuning provides the ability to move the poles (root location), thereby manipulating closed-loop system behavior.

Example 1: A true first order process without dead time, with a process gain K_P = 1 and a time constant $\tau_P = 1$, is under P-Only control. What is the impact of controller gain, K_C , on closed loop system behavior?

Solution: A first order process transfer function is $G_P(s) = \dfrac{K_P}{\tau_P s+1}$

From the problem statement, we know that

$$G_P(s)=\frac{1}{s+1}$$

The P-Only controller transfer function is

$$G_C(s) = K_C$$

Substituting $G_P(s)$ and $G_C(s)$ into the characteristic equation, we obtain

$$1 + G_C(s)G_P(s) = 1 + \frac{K_C}{s+1} = 0$$

Rearranging yields $\quad s + 1 + K_C = 0$

We recall that s is defined in the complex plane as $s = a + bi$, so

$$a + bi + 1 + K_C = 0 + 0i$$

Equating like real term gives us $a + 1 + K_C = 0$

and equating like imaginary terms: $bi = 0i$

We now see that the roots/poles as a function of K_C are

K_C	$a = -K_C - 1$	$b = 0$
0	–1	0
10	–11	0
100	–101	0

We can examine this result on the s plane of Fig. 16.4 and note that the single root always lies on the real axis as long as $K_C \geq -1$. For increasing positive values of K_C, the real root becomes increasingly negative:

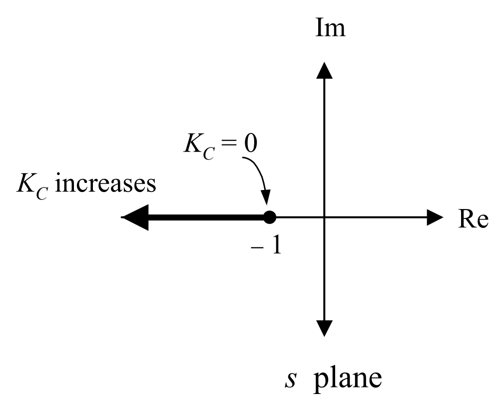

Figure 16.4 – P-Only root locus (root location) in the complex plane

All positive values of controller gain, K_C, yield a solution with no imaginary part. Hence, a true first order system under P-Only control cannot be made to oscillate, no matter how large a K_C value used. It is also unconditionally stable for all positive K_C because the root always remains on the left hand side of the s plane.

It is interesting to note that a true first order system can remain stable even when the controller gain has the wrong sign. For example, if $K_C = -0.5$, then $a = -0.5$ and $b = 0$. This root is located on the left hand side of the s plane, and thus, the system will remain stable (though control would be poor). A value of $K_C = -10$ yields an $a = 9.0$ and $b = 0$, which produces a root located on the right hand side of the s plane, indicating that the system is unstable. As the next example illustrates, even a small value of process dead time dramatically changes the inherent dynamic nature of a closed loop system.

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Example 2: A first order plus dead time (FOPDT) process with a process gain, $K_P = 1$, a time constant, $\tau_P = 1$, and a dead time, $\theta_P = 0.1$, is under P-Only control. What is the impact of controller gain, K_C, on closed loop system behavior?

Solution: A FOPDT process transfer function is $G_P(s) \; = \; \dfrac{K_P e^{-\theta_P s}}{\tau_P s + 1}$.

From the problem statement, we know that

$$G_P(s) = \frac{e^{-0.1s}}{s+1}$$

The P-Only controller transfer function is

$$G_C(s) \; = \; K_C$$

Substituting $G_P(s)$ and $G_C(s)$ into the characteristic equation, we obtain

$$1 + G_C(s)G_P(s) \; = 1 + \frac{K_C e^{-0.1s}}{s+1} \; = 0$$

We can then employ the Padé approximation:

$$e^{-0.1s} \; = \; \frac{2 - 0.1s}{2 + 0.1s}$$

Substituting the Padé approximation into the characteristic equation gives us

$$1 + \left(\frac{2 - 0.1s}{2 + 0.1s}\right)\frac{K_C}{s+1} \; = \; 0$$

Rearranging yields $\quad 0.1s^2 + (2.1 - 0.1K_C)s + 2 + 2K_C \; = \; 0$

We can then multiply both sides by 10 and then solve for the roots of the characteristic equation:

$$p_1, p_2 = \frac{-(21-K_C) \pm \sqrt{(21-K_C)^2 - 4(20+20K_C)}}{2}$$

$$= \frac{-21+K_C \pm \sqrt{(441-42K_C+K_C{}^2-80-80K_C)}}{2}$$

When the roots are real:

$$= \frac{-21+K_C \pm \sqrt{K_C{}^2 - 122K_C + 361}}{2}$$

When they have imaginary parts:

$$= \frac{-21+K_C \pm i\sqrt{-K_C{}^2 + 122K_C - 361}}{2}$$

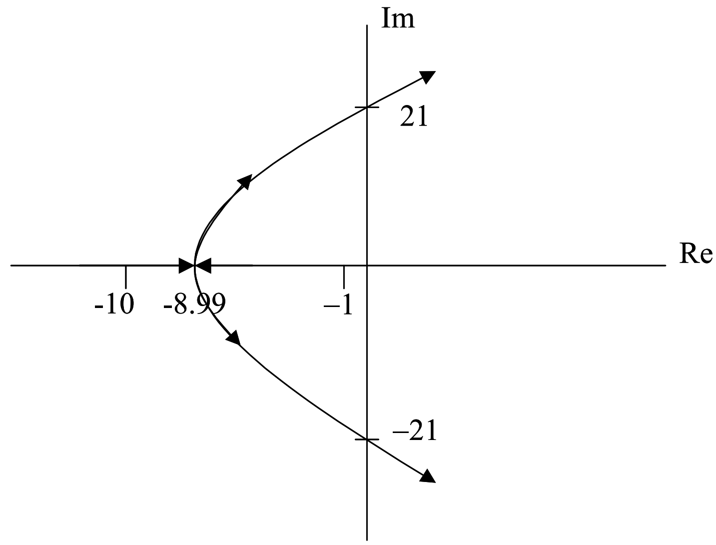
We can now solve for the roots using various values of K_C:

	K_C	P_1	P_2
	0	–1.0	–20
	1	–2.25	–17
	2	–4.0	–15
	3	–8.0	–10
repeated real roots →	3.0345	–8.99	–8.99
	3.04	$-8.98 + 0.4i$	$-8.98 - 0.4i$
	3.2	$-8.90 + 2.19i$	$-8.90 - 2.19i$
	4.0	$-8.50 + 5.27i$	$-8.50 - 5.27i$
	10.0	$-5.50 + 13.8i$	$-5.50 - 13.8i$
	20.0	$-0.50 + 20.49i$	$-0.50 - 20.49i$
limit of stability →	21.0	$0 + 20.98i$	$0 - 20.98i$
	25.0	$2 + 22.72i$	$2 - 22.72i$
	50.0	$14.5 + 28.46i$	$14.5 - 28.46i$

The limit of stability is the point where the roots fall directly on the imaginary axis (the real part of the root is zero). This is considered the limit of stability because as soon as the roots cross over to the positive real part of the *s* plane, the system becomes unstable.

An important observation from this example is that the addition of a small amount of process dead time is enough to transform a process that will not even oscillate (as shown in Example 1) into a process that will oscillate and then go unstable as controller gain, K_C, increases.

To gauge the accuracy of the Padé approximation, we could construct this problem in *Custom Process*. There we will find that the limit of stability for this system is actually closer to a controller gain $K_C = 16$ rather than the $K_C = 21$ predicted from the above anlaysis. The difference arises because Loop Pro does not employ an approximation for dead time in its calculations.



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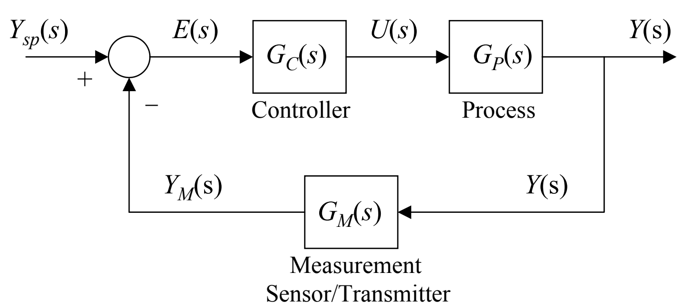
16.7 Exercises

Q-16.1 Showing all steps, derive the closed loop "set point to measured process variable" transfer function for this block diagram.



Q-16.2 Draw and label the block diagram for this process. Please use the notation given.