Filling-up and Overflow in a Tank Example

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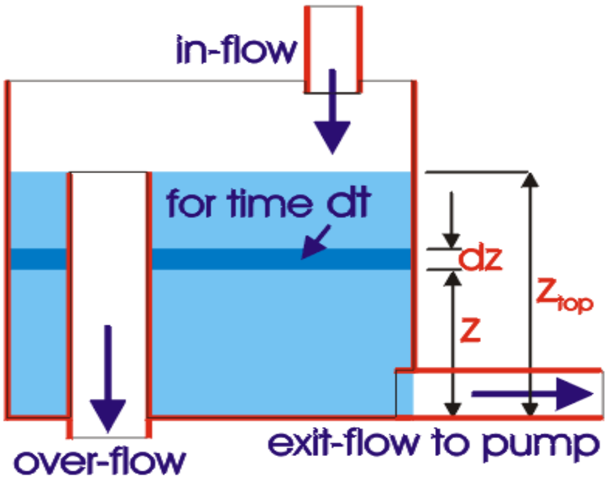
NOTE: This is **a typical, general type problem**, for which no specific prerequisites are necessary. An intelligent man may solve it without using calculus, see the last methods (c) and (d) on page 3. This is a simple problem solved as **an example to show how several different methods may be used in general.**



Problem: A tank providing water to a pump is shown in Figure. Water enters the tank trough a 1-in diameter supply pipe at a **constant** mass flow rate of 6.8 lb/s ($\mathbf{m_{in}}$), and exits to the pump through a 1-in diameter (d_e) pipe. The diameter of the tank is 18 in. ($\mathbf{d_T}$), and the top of the 2-in diameter ($\mathbf{d_{of}}$) overflow pipe is 2 ft ($\mathbf{z_{top}}$) from the base of the tank. The velocity V_e, in ft/s, of the water exiting to the pump **varies** with the height z of the water surface from the bottom, in ft, according to: $\mathbf{V_e(z)= 8.16z^{1/2}}$. **Determine the time (T)**, in seconds, for the control volume enclosing the initially empty tank **to reach steady state** (when overflow begins, $\mathbf{z=z_{top}}$). At steady state, what is the **mass flow rate**, in lb/s, that water spills **out through the overflow pipe**?

Setting up the problem:

Given: $d_e := \frac{1}{12}$ $\quad d_T := \frac{18}{12}$ $\quad d_{of} := \frac{2}{12}$ $\quad z_{top} := 2$ $\quad V_e(z) := 8.16\cdot\sqrt{z}$, also: $\rho := 62.34$

$A_e := d_e^2\cdot\frac{\pi}{4}$ $\quad A_e = 5.454\cdot10^{-3}$ $\quad A_T := \left(d_T^2 - d_{of}^2\right)\cdot\frac{\pi}{4}$ $\quad A_T = 1.745$ ... exit-pipe and tank-base areas, respectively

$m_{in} := 6.8$ ... **constant** in-flow mass rate $\quad m_e(z) := \rho\cdot A_e\cdot V_e(z)$ .. **variable** exit-flow mass rate

Conservation of mass: "small" mass (**dm**) and height (**dz**) increases for a small time period (**dt**), see the darker shaded area on the Figure above:

$$dm = \rho\cdot A_T\cdot dz = \left(m_{in} - m_e(z)\right)\cdot dt$$

...then, $D = \frac{dz}{dt} = \frac{\left(m_{in} - m_e(z)\right)}{\left(\rho\cdot A_T\right)} = C_1 - C_2\cdot\sqrt{z}$ where: $C_1 := \frac{m_{in}}{\rho\cdot A_T}$ $\quad C_1 = 0.0625$ $\quad C_2 := \frac{A_e}{A_T}\cdot 8.16$ $\quad C_2 = 0.0255$

a) Solving the above differential equation for "dt" by separation of variables and integrating:

...MathCAD numerical integration $\quad z_{top} = 2$ $\quad$ ...MathCAD symbolic/analitic integration

$$T_a := \int_0^{z_{top}} \frac{1}{\left(C_1 - C_2\cdot\sqrt{z}\right)}\,dz \qquad T_a = 54.479$$

$$\int_0^{z_{top}} \frac{1}{\left(C_1 - C_2\cdot\sqrt{z}\right)}\,dz \rightarrow 54.4789751206557729$$

...the solution, i.e. the time needed to fill-up the empty tank till its overflow level.

NOTE: Different subscripts may be used for the same physical variables to indicate the corresponding method, for example (T_a) or (T_b) or (T_I), or to differentiate results obtained using different methods.

Filling-up and Overflow in a Tank (Cont'd)

$$I(z) := \frac{-\ln\left(C_1^{\,2} - C_2^{\,2}\cdot z\right)}{C_2^{\,2}}\cdot C_1 - 2\cdot\frac{\sqrt{z}}{C_2} + 2\cdot\frac{C_1^{\,2}}{\left[C_2\cdot\left(\sqrt{-C_2^{\,2}}\cdot\sqrt{C_1^{\,2}}\right)\right]}\cdot \operatorname{atan}\left(\sqrt{-C_2^{\,2}}\cdot\frac{\sqrt{z}}{\sqrt{C_1^{\,2}}}\right)$$

...the analytical indefinite integral of the above expression

Also, $T_I := I(z_{top}) - I(0)$ $T_I = 54.479$ ...the solution, i.e. the time needed to fill-up the empty tank till its overflow level.

As we often see, the **analytical integration is usually lengthy <u>or impossible</u>**, so **we, engineers, resort to numerical solutions**. That is no longer difficult, if we utilize available (and now powerful and inexpensive) PC computational hardware and application software, see below.

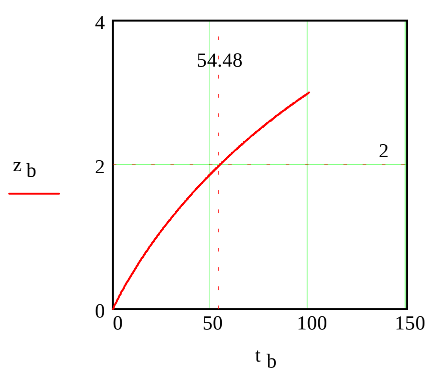
b) Solving the above differential equation (see page 1) by *Runge-Kutta* numerical method (with *rkfixed*( ), a built-in MathCAD function - have to be used if the separation of variables is not possible):

$z_0 := 0$ $D(t, z) := C_1 - C_2\cdot\sqrt{z_0}$ ...initial value and derivative of the function **z**(t)

$t_0 := 0$ $t_L := 100$ $N_1 := 500$ ...initial- and end-values of the independent variable **t** and number of points over that range

$Z := \text{rkfixed}(z, t_0, t_L, N_1 - 1, D)$ $N := \text{rows}(Z)$ $N = 500$ $M := \text{cols}(Z)$ $M = 2$ $\left(Z^{\langle 1\rangle}\right)_{N-1} = 3.001256$

$t_b := Z^{\langle 0\rangle}$ $z_b := Z^{\langle 1\rangle}$ ...the solution of the above differential equation (see page 1) over the range $t_0<t<t_L$.



The Runge-Kutta method (see also MathCAD Help, or press F1 key while cursor is on "*rkfixed*" function) is a very effective to solve any first-order ordinary differential equation (ODE), or a system of the first order ODEs (thus higher-order DE) for a given initial value(s). However, to evaluate the independent variable, t in this case, for a given value of dependent variable $z=z_{top}=2$, we have to "look" in the table of solution values, or on the diagram, or to try different t-values until we obtain the given z=2 value, in this case. Luckily, MathCAD has the built-in **"Given-Find" solver** to do almost any iterative "search" for us, see next (**This procedure may be also used to solve the so-called boundary-value problems using the initial value problem methods**):

Define the solution as function of t_{top}: $Z_b(t_{top}) := \text{rkfixed}(z, t_0, t_{top}, N-1, D)$ $z_{Top}(t_{top}) := \left(Z_b(t_{top})^{\langle 1\rangle}\right)_{N-1}$

Guess: $t_{top} := 40$ ...and use the "Given-Find" solver below to find new $T_b=t_{top}$ to satisfy the required condition

Given

$z_{Top}(t_{top}) = 2$ ... given condition

$T_b := \text{Find}(t_{top})$ $T_b = 54.479$ ...the solution, i.e. the time needed to fill-up the empty tank till its overflow level.

Check the z-value for the calculated T_b value, i.e. that the condition above is satisfied: $z_{Top}(T_b) = 2$

c) Solving the above differential equation (see page 1) by dividing the function z-range in N divisions and using the finite difference approximation (this is a "common sense" method):

$N := 100$ $\quad z_{top} := 2$ $\quad \Delta z := \frac{z_{top}}{N}$ $\quad i := 0..N-1$ $\quad t_0 := 0$

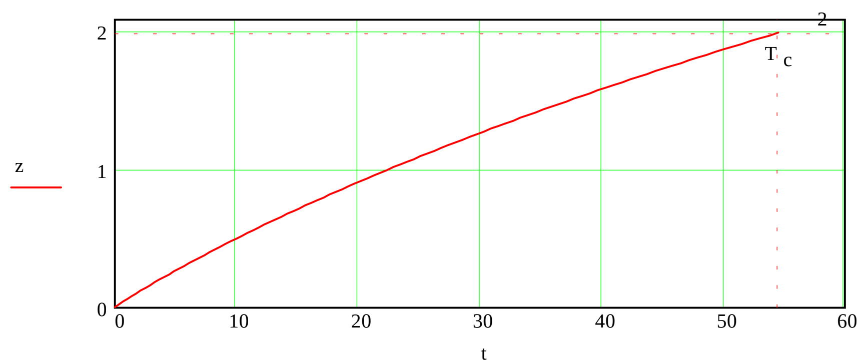
$z_i := \frac{i}{N} \cdot z_{top}$ $\quad z_N := z_{top}$ $\quad z_{av_i} := \frac{z_i + z_{i+1}}{2}$ $\quad \Delta t_i := \frac{\Delta z}{C_1 - C_2 \cdot \sqrt{z_{av_i}}}$ $\quad t_{i+1} := \sum_{k=0}^{i} \Delta t_k$

$K := \text{if}(N < 12, N, 12)$

$K = 12$ $\quad T_c := \sum \Delta t$ $\quad T_c = 54.48$ $\quad$ also, $\quad t_N = 54.48$

$k := 0..K$ $\quad$ ...the solution, i.e. the time needed to fill-up the empty tank till its overflow level.

$t_k =$	$z_k =$
0	0
0.334	0.02
0.678	0.04
1.03	0.06
1.389	0.08
1.753	0.1
2.124	0.12
2.499	0.14
2.879	0.16
3.264	0.18
3.653	0.2
4.046	0.22
4.444	0.24



...first K-values tabulated on the right and all values plotted above.

d) Solving the above problem (see page 1) by approximating variable exit flow rate with a constant flow rate based on average water height during the fill-up

NOTE: This is a "common sense" simplified method which may or may not be appropriate for a given case. Be very cautious when using it.

$z_{avg} := \frac{0 + z_{top}}{2}$ $\quad m_{e\_av} := m_e(z_{avg})$ $\quad T_d := \frac{\rho \cdot A_T \cdot z_{top}}{m_{in} - m_{e\_av}}$ $\quad T_d = 54.057$ $\quad \frac{T_d}{T_I} = 99.2\%$

The T_d compares pretty well with the above results (T_a, T_I, T_b, T_c), since the relation between z and t is "almost" linear (see above diagram). However, we have to be very cautious in general when using such approximations.

The steady-state solution (at $z = z_{top} = 2 =$ constant):

$m_{in} = 6.8$ ... in-flow mass rate

$m_{e\_top} := m_e(z_{top})$ $\quad m_{e\_top} = 3.924$ ... exit-flow mass rate

$m_{overflow} := m_{in} - m_{e\_top}$ $\quad m_{overflow} = 2.876$ ... over-flow mass rate